Twindragon

Represented in a quaternion space

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Number Systems

A **number system** consist of base a b and a set of digits D.

Digit is a single symbol used to make a numeral.

Base is the number that gets multiplied when using an exponent.

A **fraction** in number system is an expression of the form:

$$(0.d_1d_2d_3...)_b = \sum_{n=1}^{\infty} d_n b^{-n}$$

where each $d_n \in D$.

Decimal system

The **decimal system** is the most commonly used number system in our everyday life. In such a system our base $b = 10$ and the set of digits for such a system is $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

For example a number 0.357 can be represented by the digits $d_0 = 3, d_1 = 5$, and $d_2 = 7$. As our base $b = 10$, we get:

$$\begin{aligned}(0.d_1d_2d_3)_{10} &= 3 \cdot 10^{-1} + 5 \cdot 10^{-2} + 7 \cdot 10^{-3} \\ &= 0.357\end{aligned}$$

The **standard binary number system** has base $b = 2$ and the set of digits $D = \{0, 1\}$.

Let's consider the binary number $(0.01)_b$. Here in the base $b = 10$ we have:

$$\begin{aligned}(0.01)_{10} &= 0 \cdot 10^{-1} + 1 \cdot 10^{-2} \\ (0.01)_{10} &= 10^{-2}\end{aligned}$$

In a base 2, we have:

$$\begin{aligned}(0.01)_2 &= 0 \cdot 2^{-1} + 1 \cdot 2{-}2 \\ (0.01)_2 &= 2^{-2}\end{aligned}$$

There is alternative binary number system that was discovered by computer scientist **Donald Knuth** and mathematician **Christopher Davis** in 1970 (see [2]). In this binary system the base is the complex number $b = -1 + i$ and the set of digits $D = \{0, 1\}$. In this system:

$$\begin{aligned}(0.01)_{-1+i} &= 0 \cdot (-1+i)^{-1} + 1 \cdot (-1+i)^{-2} \\ (0.01)_{-1+i} &= \frac{1}{2}i\end{aligned}$$

For the rest of this paper we fix $b = -1 + i$.

Twindragon

Plotting all fractions in the complex plane using base $b = -1 + i$ and digits $D = \{0, 1\}$ gives the following:

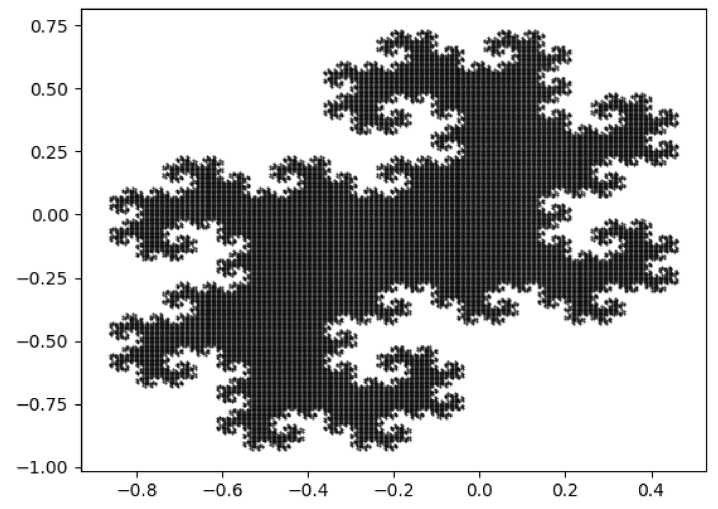

Figure 1: The twindragon in the complex plane

The region above is called a **fractal**. The boundary of the figure is an example of a snowflake curve.This snowflake curve is a fractal with dimension that is approximately 1.5236 (see [1]). Two twindragons of the same size and order can be connected by joining the head of each to the tail of the other. Similarly, we can use translation (shift the figure) by adding any complex number. Thus, twindragons tile the complex plane. Here is an example of tiling four twindragons together.

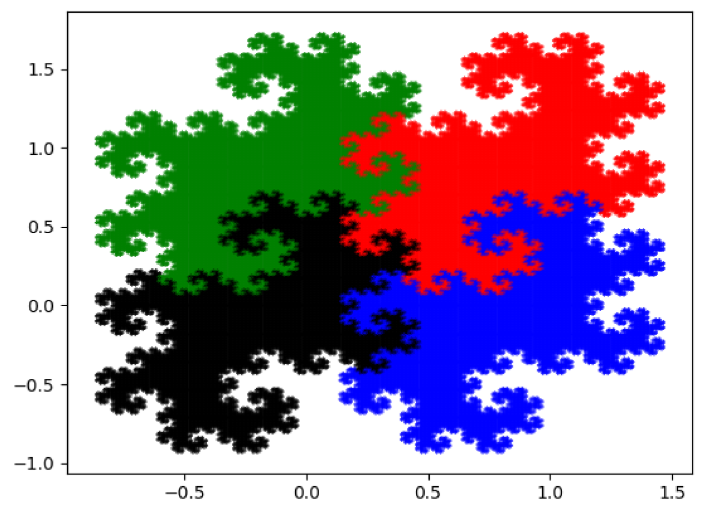

Figure 2: twindragons tiling the plane

A New Quaternion Number System

The quaternions form an algebra that extends the complex numbers. They were first described by Irish mathematician William Rowan Hamilton in 1843 and applied to mechanics in three-dimensional space. The space of quaternions is 4-dimensional with each quaternion having the form:

$$\alpha + \beta i + \gamma j + \delta k$$

Here $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and i, j, k are units satisfying the famous equations:

$$i^2 = j^2 = k^2 = ijk = -1$$

We considered the number system with $b = -1 + i$ and the set of digits $D = \{0, 1, k\}$. For example:

$$(0.0k)_b = k \cdot b^{-2} = k \cdot (\frac{1}{2}i) = \frac{1}{2}j$$

$$\begin{aligned}(0.k1)_b &= k \cdot b^{-1} + 1 \cdot b^{-2} \\ &= k \cdot (\frac{-1-i}{2}) + \frac{i}{2} \\ &= \frac{1}{2}i - \frac{1}{2}j - \frac{1}{2}k\end{aligned}$$

Fractions in our Quaternion Number System

As we use only the set of digits D and the base b, we can get an i part from the base and j part by multiplying i part by k. Note that fractions in this system live in 4-dimensional quaternion space. Here are some 2-dimensional slices of the set of fractions.

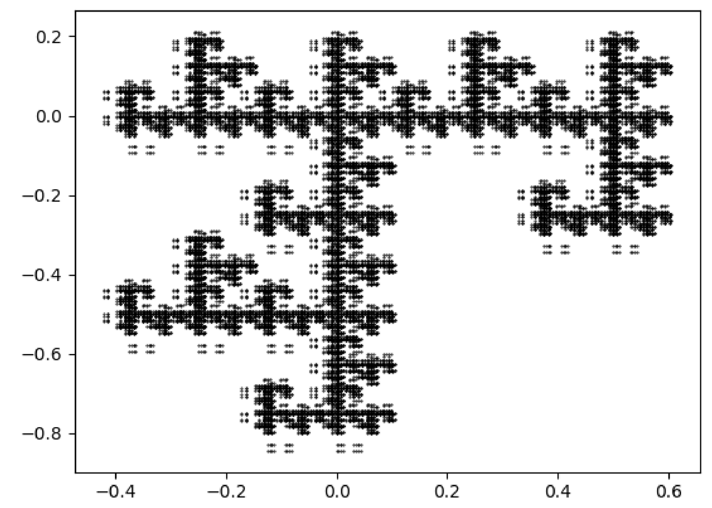

Figure 3: Fractions in the i,k-plane

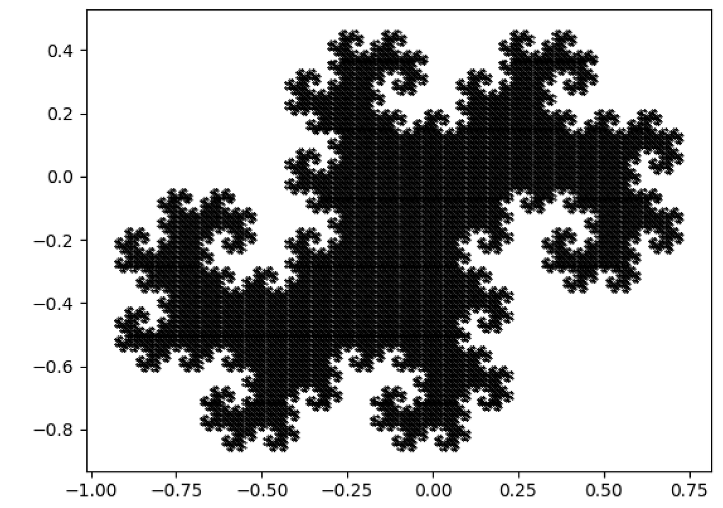

Figure 4: Fractions in the j,k-plane

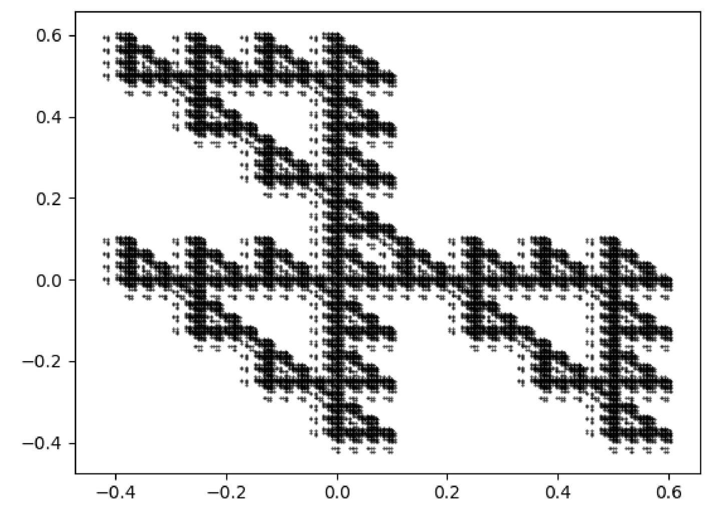

Figure 5: Fractions in the i,j - plane

The structure in Figure 1 is the same as the structure in Figure 4, however, it is inverted. When you multiply k by any number of the form $(0.d_1d_2d_3d_4...)_b$, where $d_n \in \{0, 1\}$, the i part becomes the j part and the $Real$ part becomes the k part.

However, that is not the only figures that can be obtained by this structure. Let's look at the plane $P_0 = \{\frac{1}{2}i + \beta j + \gamma k \mid \beta, \gamma \in \mathbb{R}\}$. The structure that we get is graphically represented by:

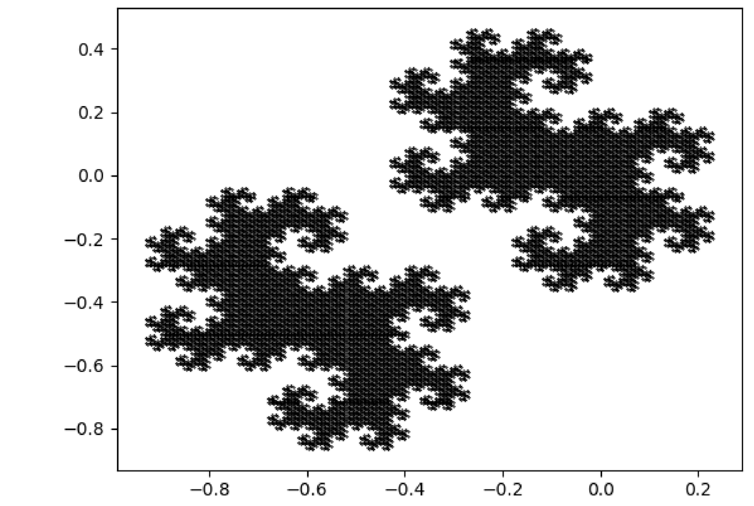

Figure 6: Two mini twindragons

As we discussed earlier $(0.01)_b = \frac{1}{2}i$. In this case, for P_0 we fix the digit $d_2 = 1$, thus, all the numbers $(0.d_10d_3d_4...)_b$ or $(0.d_1kd_3d_4...)_b$ do not lie in P_0. Therefore, we can see 2 separate twindragons. As the plane P_0 exists in i, j, k, where the i part is represented by $d_2 = 1$ only. Thus, we have two different collections of fractions where one is represented by $(0.01d_3d_4d_5...)_b$ and the other collection is represented by $(0.k1d_3d_4d_5...)_b$. As there is only 2 different collections of fractions we can see two mini-twindragons in P_0.

Similarly, if we look at the plane $P_1 = \{-\frac{1}{8}i + \beta j + \gamma k | \beta, \gamma \in \mathbb{R}\}$, we will get the graphic representation of our structure such as:

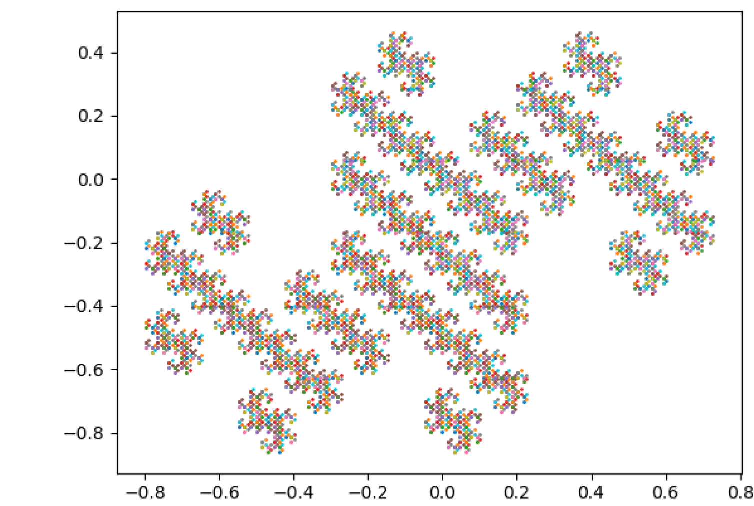

Figure 7: 32 mini twindragons

For this representation we fix the digit $d_6 = 1$, since $(0.000001)_b = -\frac{1}{8}i$. We obtain 32 mini dragons for the same reason as in Figure 6. In fact, we can obtain the string represented only by imaginary part if and only if $d_{2+4m} = 1$ and all the other digits equal to 0. If we fix $d_{2+4m} = 1$ in P_m we get 2^{1+4m} twindragons. At the same time, we can see that the shape of the Figure 7 almost represents the original twindragon. This is true as if $m \to \infty$ we get the original twindragon. As our structure represent the collection of all the fractions in complex space, it means that the twindragon will not dilate if we fix a digit, thus, we need to fit all the twindragons into the same area as the original twindragon. As $m \to \infty$ the amount of twindragons is $\lim_{m\to\infty} 2^{1+4m} = \infty$ and all of our mini-twindragons obtained will fill the space of the original twindragon.

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References

[1] Benoit B. Mandelbrot. *The Fractal Geometry of Nature*. W. H. Freeman Company, January 1, 1982.

[2] Christopher Davis and Donald Knuth. *Number representations and dragon curves*. J. of Recreational Mathematics, 366–81, 133–149, 1970.

[3] Gerald Edgar. *Measure Topology and Fractal Geometry*. Springer 2nd Edition 2008.